Most central or least central?
How much modeling decisions influence a node’s centrality ranking in multiplex networks

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1 Multiplex Networks
- Degree Centrality in Multiplex networks

2 Modeling decisions
- Different Normalization Strategies
- Aggregation Strategies

3 The sensitivity of nodes to modeling decisions
- European airlines data set
- $\Delta_{norm}$ and $\Delta_{agg}$
- Higgs Boson Tweets Network
- Law Firm dataset
Understanding of Complex systems needs more complicated frameworks like multiplex networks than ordinary simple graphs [7].
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Analyzing the influence of nodes in networks –from simple graphs to multilayer networks– is always a fundamental question to be addressed in order to solve many real problems [6].
Analyzing flow processes in multiplex networks such as epidemic transmission in Transportation networks [2, 4].
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Identifying cancer drivers in Biological networks using the representation of protein-protein interaction, gene regulation, co-expression, and metabolic network in a multiplex network [1].
The usefulness of Centrality Measures in Multiplex Networks

- Analyzing flow processes in multiplex networks such as epidemic transmission in Transportation networks [2, 4].
- Identifying cancer drivers in Biological networks using the representation of protein-protein interaction, gene regulation, co-expression, and metabolic network in a multiplex network [1].
- Analyzing leading drivers in Terrorist networks, where for instance, the importance of a node in “communication” layer is affected by the importance of the node in “trust” layer [6].
Degree Centrality as the simplest index in Multiplex networks

- A network with $|\mathcal{L}|$ layers
  $\mathcal{L} = \{L_1, L_2, \cdots, L_{|\mathcal{L}|}\}$ where each layer $L_i$ is a simple graph comprised of a set of $V_i$ nodes and $E_i \subseteq V_i \times V_i$ edges.

- A set of nodes are common:
  $V^* = \bigcap_{i=1}^{|\mathcal{L}|} V_i$.

- The degree $\deg_i(v)$ of any node $v$ is defined as the number of edges connected to the node $v$ in layer $L_i$.

- The result of ranking is from position 1 to position $|V^*|$.
Given multiple layers with different structure and the different position of nodes in layers, obtaining a single ranking of centrality is more complicated.

A comparison of centrality index values of nodes in different layers requires a careful normalization before the aggregation.
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Even the most simplest index using different modeling decisions can turn a node from the most central to the least central!
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**Different modeling decisions**

The normalization strategies

**NormMethod 1**, for layer $L_i$ takes $\text{deg}_i(v)$ for all $v \in V^*$ and normalizes it with the minimum and maximum values in the set of common nodes. This results in a vector of normalized indices of $[0, 1]$ for layer $L_i$.

$$C_1(v, i) = \frac{\text{deg}_i(v) - \min\{\text{deg}_i(v) | v \in V^*\}}{\max\{\text{deg}_i(v) | v \in V^*\} - \min\{\text{deg}_i(v) | v \in V^*\}}$$
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**NormMethod 2** is similar to the last method but the normalization is done using the minimum and maximum values in the set of all nodes ($V_i$) in layer $L_i$.

$$C_2(v, i) = \frac{\text{deg}_i(v) - \min\{\text{deg}_i(v) | v \in V_i\}}{\max\{\text{deg}_i(v) | v \in V_i\} - \min\{\text{deg}_i(v) | v \in V_i\}}$$
**NormMethod 3** uses the results by *NormMethod 2* and multiplies them with the fraction of the maximum degree in layer $L_i$ and the maximum degree among all nodes in all $|\mathcal{L}|$ layers. This results in a vector of indices of nodes ($v \in V_i$) between $[0, \frac{\max\{\deg_i(v) | v \in V_i\}}{\max\{\deg_i(v) | v \in \bigcup V_j, 1 \leq i \leq |\mathcal{L}|\}}]$.

\[
C_3(v, i) = C_2(v) \cdot \left( \frac{\max\{\deg_i(v) | v \in V_i\}}{\max\{\deg_i(v) | v \in \bigcup V_j, i \in [1, \ldots, |\mathcal{L}|]\}} \right)
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The normalization strategies...

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**NormMethod 4** for each layer, we rank the nodes non-increasingly by their degree $\deg_i(v)$ and obtain $r_i(v)$. This is then normalized by $n_i$.

$$C_4(v, i) = \frac{r_i(v)}{n_i}$$
**Why the normalized ranking?**

**Figure:** As can be seen, 90% of the degrees in Lufthansa are smaller than 70% of the degrees in Airberlin. If an aggregation wants to reward at least the most central nodes on each layer, this is difficult as even medium central nodes in Airberlin would get a larger index than most of the Lufthansa nodes.
Maximum Entropy Ordered Weighted Averaging (MEOWA) operator (denoted by \( \lambda \)) creates a single number based on the vector of a node’s \(|\mathcal{L}|\) normalized degrees as follows:

\[
\lambda(C_x(v, 1), C_x(v, 2), \cdots, C_x(v, |\mathcal{L}|)) = \sum_j w_j \ d_j(v)
\]

where \( D = (b_1, b_2, \ldots, b_{|\mathcal{L}|} \) is the non-increasingly sorted vector of the normalized degrees, and \( w \) is a weight vector. The weight vector is obtained using the following function based on a parameter \( \beta \) [5]:

\[
w_i = \frac{e^{\beta \frac{n-i}{n-1}}}{\sum_{j=1}^{n} e^{\beta \frac{n-j}{n-1}}}.
\]
The aggregation strategies...

- $\beta = 20$: the weight vector is close to $(1, 0, \ldots, 0)$ and the aggregation strategy is $(OR$-operator); at least one layer.
- $\beta = -20$: the weight vector is close to $(0, 0, \ldots, 1)$ and the aggregation strategy is $(AND$-operator); all layers.
- $\beta = 0$: the weight vector is $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ and the aggregation strategy is $(Average)$.

Any $\beta$-value between the extreme strategies of “at least one” and “all layers” can be described using a set of proportional linguistic quantifiers (a few, some, most, almost introduced by Zadeh [9]).

$$\Omega = \frac{1}{n-1} \sum_{i=1}^{n} (n - i) \frac{e^{\beta \frac{n-i}{n-1}}}{\sum_{j=1}^{n} e^{\beta \frac{n-j}{n-1}}}$$
Outline

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Sude Tavassoli (TU KL)
European airlines data set

A network comprised of four layers of airlines: Air Berlin, Easyjet, Lufthansa, and Ryan air. The order varies from 75 to 128 among four layers [2]. 9 nodes are common among the four layers.

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$\deg(Manchester) : 1, 12, 5, 5 \rightarrow C_1(v) : 0, 0.667, \boxed{1}, 0$

$\deg(Francisco) : 12, 5, 1, 15 \rightarrow C_1(v) : 0.44, 0.2, 0, 0.435$
EUROPEAN AIRLINES DATA SET

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### European airlines data set

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$C_4(v) : 0.093, 0.818, \boxed{0.887}, 0.461$

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$C_4(v) : \boxed{0.833}, 0.611, 0.184, 0.789$
Measuring the sensitivity of the nodes to the modeling decisions

The overall sensitivity of a node on the chosen normalization strategy is:

$$\Delta_{\text{norm}}(v) := \max \{ \max \text{Rank}(v, \beta) - \min \text{Rank}(v, \beta) | \beta \in \Gamma \}$$

where $\Gamma$ is a set of different $\beta$-values.
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**Example:**

\[
\Delta \text{norm}(Manchester) := \max \{3, 3, 2, 2, 1, 3, 4, 5, 5, 5, 5, 5\} = 5
\]
Measuring the sensitivity of the nodes to the modeling decisions

Let $minRank(v, C_i)$ denote the minimal rank of node $v$ based on normalization strategy $C_i$ over all $\beta$-values and define $maxRank(v, C_i)$ accordingly.

The overall sensitivity of a node on the chosen aggregation strategy is:

$$\Delta_{agg}(v) := \max\{maxRank(v, C_i) − minRank(v, C_i) | 1 \leq i \leq 4\}$$
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**Example:**

$$\Delta_{agg}(Manchester) := \max\{5, 2, 2, 5\} = 5$$
Now if we remove the layer of Lufthansa from the aggregation scenario, then we have 20 common nodes.

**Figure:** The sensitivity of the 20 common airports to the choice of aggregation strategy and normalization strategy. The four sections contain the nodes sensitive to only one choice (A0N+ or A + N0), those sensitive to none (A0N0), or both (A + N+).
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**Tweets dataset**

A network comprised of four layers representing different interactions on topics concerning the “Higgs Boson”: *mentioning*, *replying* to the tweets, *re-tweeting* the tweets of the other users, plus the social network of followers/followees [3].

**Figure:** The sensitivity of 127 common nodes among four layers to the choices of aggregation ($\Delta \text{agg}$) and normalization ($\Delta \text{norm}$).

**Figure:** The ranking positions obtained using the different aggregation strategies (using the $\beta$ parameter) for the aggregation of the four layers.
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A network comprised of three layers of seeking advice, co-working, and having a friendship outside the firm among 71 attorneys [8].

**Figure**: The sensitivity of 71 nodes to the choices of different aggregation strategies ($\Delta_{agg}$) and the different normalization methods ($\Delta_{norm}$).

**Figure**: The rankings obtained using the different aggregation strategies (using the $\beta$ parameter) for the aggregation of the results of three layers.
The results show that even a seemingly simple measure using different models can turn a node from being the most central to the least central.
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The observed high sensitivity of single nodes to the specific choice of aggregation and normalization strategies emphasizes that all these models are of strong importance, especially for all kinds of intelligence-analytic software, as it questions the interpretations of the findings.

All these preprocessing steps need to be documented to make the analysis reproducible and its interpretation analyzable.
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Future works: categorizing nodes using fuzzy linguistic terms with respect to their overall centrality index.
References


